43. (e) The y-intercept of the derivative is $b - a$.

44. Since the function must be continuous at $x = 1$, we have
$\lim_{x\to 1^+}(3x+k) = f(1) = 1$, so $3+k=1$, or $k=-2$.

This gives $f(x) = \begin{cases} x^3, & x \le 1 \\ 3x-2, & x > 1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{(1+h)^3-(1)^3}{h}$$
$$= \lim_{h\to 0^-}\frac{3h+3h^2+h^3}{h}$$
$$= \lim_{h\to 0^-}(3+3h+h^2) = 3$$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{[3(1+h)-2]-(1)^3]}{h}$$
$$= \lim_{h\to 0^+}\frac{(1+3h)-1}{h} = \lim_{h\to 0^+} 3 = 3$$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

45. (a) $1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.992$

Alternate method: $\frac{{}_{365}P_3}{365^3} \approx 0.992$

(b) Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

(c) Let P represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is

$(1-P)\left(\frac{362}{365}\right)$, so the probability of a shared birthday is

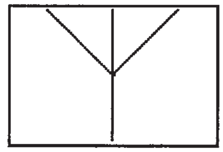
$1-(1-P)\left(\frac{362}{365}\right) \approx 0.016$.

(d) No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

Section 3.2 Differentiability (pp. 109–115)

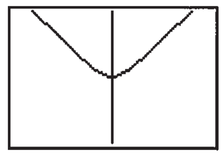
Exploration 1 Zooming in to "See" Differentiability

1. Zooming in on the graph of f at the point (0, 1) always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.



[−0.25, 0.25] by [0.836, 1.164]

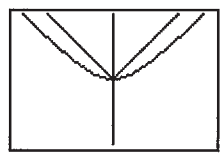
2. Zooming in on the graph of g at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window.
[−4, 4] by [−1.624, 3.624].



[−0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.



[−0.03125, 0.03125] by [0.9795, 1.0205]

Exploration 2 Looking at the Symmetric Difference Quotient Analytically

1. $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^2-10^2}{0.01} = 20.01$

$f'(10) = 2 \cdot 10 = 20$

The difference quotient is 0.01 away from $f'(10)$.

2. $\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^2-(9.99)^2}{0.02} = 20$

The symmetric difference quotient exactly equals $f'(10)$.

3. $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^3-10^3}{0.01} = 300.3001$.

$f'(10) = 3 \cdot 10^2 = 300$

The difference quotient is 0.3001 away from $f'(10)$.

$\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^3-(9.99)^3}{0.02} = 300.0001$. The symmetric difference quotient is 0.0001 away from $f'(10)$.

Quick Review 3.2

1. Yes

2. No (The $f(h)$ term in the numerator is incorrect.)

3. Yes **4.** Yes

5. No (The denominator for this expression should be $2h$).

6. All reals

7. $[0, \infty)$ **8.** $[3, \infty)$

9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.

10. $$\frac{f(3+0.001)-f(3-0.001)}{0.002} = \frac{5(3+0.001)-5(3-0.001)}{0.002} = \frac{5(0.002)}{0.002} = 5$$

Section 3.2 Exercises

1. Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2-0}{h} = \lim_{h\to 0^-} h = 0$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1$$

Since $0 \neq 1$, the function is not differentiable at the point P.

2. Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{2-2}{h} = \lim_{h\to 0^-} 0 = 0$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)-2}{h} = \lim_{h\to 0^+} 2 = 2$$

Since $0 \neq 2$, the function is not differentiable at the point P.

3. Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{\sqrt{1+h}-1}{h}$$
$$= \lim_{h\to 0^-} \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0^-} \frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0^-} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{[2(1+h)-1]-1}{h}$$
$$= \lim_{h\to 0^+} \frac{2h}{h}$$
$$= \lim_{h\to 0^+} 2 = 2$$

Since $\frac{1}{2} \neq 2$, the function is not differentiable at the point P.

4. Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)-1}{h} = \lim_{h\to 0^-} 1 = 1$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{1+h}-1}{h}$$
$$= \lim_{h\to 0^+} \frac{1-(1+h)}{h(1+h)}$$
$$= \lim_{h\to 0^+} \frac{-h}{h(1+h)}$$
$$= \lim_{h\to 0^+} -\frac{1}{1+h} = -1$$

Since $1 \neq -1$, the function is not differentiable at the point P.

5. **(a)** All points in $[-3, 2]$

(b) None

(c) None

6. **(a)** All points in $[-2, 3]$

(b) None

(c) None

7. **(a)** All points in $[-3, 3]$ except $x = 0$

(b) None

(c) $x = 0$

8. **(a)** All points in $[-2, 3]$ except $x = -1, 0, 2$

(b) $x = -1$

(c) $x = 0, x = 2$

9. **(a)** All points in $[-1, 2]$ except $x = 0$

(b) $x = 0$

(c) None

10. **(a)** All points in $[-3, 3]$ except $x = -2, 2$

(b) $x = -2, x = 2$

(c) None

11. Since $\lim_{x\to 0} \tan^{-1} x = \tan^{-1} 0 = 0 \neq y(0)$, the problem is a discontinuity.

12. $$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^{4/5}}{h} = \lim_{h\to 0^-} \frac{1}{h^{1/5}} = -\infty$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^{4/5}}{h} = \lim_{h\to 0^+} \frac{1}{h^{1/5}} = \infty$$

The problem is a cusp.

13. Note that $y = x + \sqrt{x^2} + 2 = x + |x| + 2$

$$= \begin{cases} 2, & x \le 0 \\ 2x+2, & x > 0. \end{cases}$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{2-2}{h} = \lim_{h\to 0^-} 0 = 0$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(2h+2)-2}{h} = \lim_{h\to 0^+} 2 = 2$$

The problem is a corner.

14. $$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(3-\sqrt[3]{h})-3}{h} = \lim_{h\to 0} -\frac{\sqrt[3]{h}}{h}$$

$$= \lim_{h\to 0}\left(-\frac{1}{h^{2/3}}\right) = -\infty$$

The problem is a vertical tangent.

15. Note that $y = 3x - 2|x| - 1 = \begin{cases} 5x-1, & x \le 0 \\ x-1, & x > 0 \end{cases}$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(5h-1)-(-1)}{h} = \lim_{h\to 0^-} 5 = 5$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(h-1)-(-1)}{h} = \lim_{h\to 0^+} 1 = 1$$

The problem is a corner.

16. $$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\sqrt[3]{|h|}-0}{h} = \lim_{h\to 0^-} \frac{-\sqrt[3]{h}}{h}$$

$$= \lim_{h\to 0^-}\left(-\frac{1}{h^{2/3}}\right) = -\infty$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt[3]{|h|}-0}{h} = \lim_{h\to 0^+} \frac{\sqrt[3]{h}}{h}$$

$$= \lim_{h\to 0^+} \frac{1}{h^{2/3}} = \infty$$

The problem is a cusp.

17. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{4(0.001)-(0.001)^2-(4(-0.001)-(-0.001)^2)}{0.002}$$

$= 4$, yes it is differentiable.

18. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{4(3.001)-(3.001)^2-(4(2.999)-(2.999)^2}{0.002}$$

$= -2$, yes it is differentiable.

19. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{4(1.001)+(1.001)^2-4(0.999)-(0.99)^2}{0.002}$$

$= 2$, yes it is differentiable.

20. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{(0.001)^3-4(0.001)-((-0.001)^3-4(-0.001))}{0.002}$$

$= -3.999999$, yes it is differentiable.

21. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{(-1.999)^3-4(-1.999)-((-2.001)^3-4(-2.001))}{0.002}$$

$= 8.000001$, yes it is differentiable.

22. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h}$$

$$= \frac{(2.001)^3-4(2.001)-((1.999)^3-4(1.999))}{0.002}$$

$= -8.000001$, yes it is differentiable.

23. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{(0.001)^{2/3}-(-0.001)^{2/3}}{0.002}$$

$= 0$, no it is not differentiable. (CUSP)

24. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{|3.001-3|-|2.999-3|}{0.002}$$

$= 0$, no it is not differentiable. (CORNER)

25. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

$$= \frac{(0.001)^{2/5}-(-0.001)^{2/5}}{0.002}$$

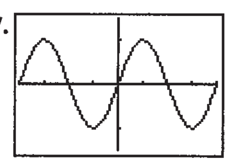
$= 0$, no it is not differentiable. (CUSP)

26. $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h}$$

$$= \frac{(0.001)^{4/5}-(-0.001)^{4/5}}{0.002}$$

$= 0$, no it is not differentiable. (CUSP)

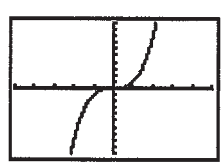
27.



$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

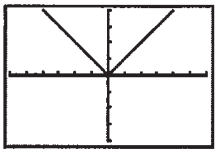
$$\frac{dy}{dx} = \sin x$$

28.



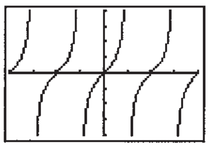
[−5, 5] by [−10, 10]

$$\frac{dy}{dx} = x^3$$

29. 

[−6, 6] by [−4, 4]

$\frac{dy}{dx} = \text{abs}(x)$ or $|x|$

30. 

[−2π, 2π] by [−4, 4]

$\frac{dy}{dx} = \tan x$

Note: Due to the way NDER is defined, the graph of $y = \text{NDER}(x)$ actually has two asymptotes for each asymptote of $y = \tan x$. The asymptotes of $y = \text{NDER}(x)$ occur at $x = \frac{\pi}{2} + k\pi \pm 0.001$, where k is an integer. A good window for viewing this behavior is [1.566, 1.576] by [−1000, 1000].

31. Find the zeros of the denominator.

$$x^2 - 4x - 5 = 0$$
$$(x+1)(x-5) = 0$$
$$x = -1 \text{ or } x = 5$$

The function is a rational function, so it is differentiabe for all x in its domain: all reals except $x = -1, 5$.

32. The function is differentiable except possibly where $3x - 6 = 0$, that is, at $x = 2$. We check for differentiability at $x = 2$, using k instead of the usual h, in order to avoid confusion with the function $h(x)$.

$$\lim_{k\to 0} \frac{h(2+k) - h(2)}{k} = \lim_{k\to 0} \frac{[\sqrt[3]{3(2+k) - 6} + 5] - 5}{k}$$
$$= \lim_{k\to 0} \frac{\sqrt[3]{3k}}{k} = \sqrt[3]{3} \lim_{k\to 0} \frac{1}{k^{2/3}} = \infty$$

The function has a vertical tangent at $x = 2$. It is differentiable for all reals except $x = 2$.

33. Note that the sine function is odd, so

$$P(x) = \sin(|x|) - 1 = \begin{cases} -\sin x - 1, & x < 0 \\ \sin x - 1, & x \ge 0. \end{cases}$$

The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

34. Since the cosine function is even, $Q(x) = 3\cos(|x|)$ $= 3\cos x$. The function is differentiable for all reals.

35. The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)^2 - 1}{h} = \lim_{h\to 0^-} \frac{h^2 + 2h}{h}$$
$$= \lim_{h\to 0^-} (h+2) = 2$$
$$\lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^+} \frac{(2h+1) - 1}{h} = \lim_{h\to 0^+} 2 = 2$$

The function is differentiable at $x = 0$.

Check $x = 3$:
Since $g(3) = (4 - 3)^2 = 1$ and $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (2x+1) = 2(3) + 1 = 7$, the function is not continuous (and hence not differentiable) at $x = 3$. The function is differentiable for all reals except $x = 3$.

36. Note that $C(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$, so it is differentiable for all x except possibly at $x = 0$.

Check $x = 0$:

$$\lim_{h\to 0} \frac{C(0+h) - C(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0$$

The function is differentiable for all reals.

37. The function $f(x)$ does not have the intermediate value property. Choose some a in (−1, 0) and b in (0, 1). Then $f(a) = 0$ and $f(b) = 1$, but f does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, f cannot be the derivative of any function on [−1, 1].

38. **(a)** $x = 0$ is not in their domains, or, they are both discontinuous at $x = 0$.

(b) For $\frac{1}{x}$: $\text{NDER}\left(\frac{1}{x}, 0\right) = 1{,}000{,}000$

For $\frac{1}{x^2}$: $\text{NDER}\left(\frac{1}{x^2}, 0\right) = 0$

(c) It returns an incorrect response because even though these functions are not defined at $x = 0$, they are defined at $x = \pm 0.001$. The responses differ from each other because $\frac{1}{x^2}$ is even (which automatically makes NDER $\left(\frac{1}{x^2}, 0\right) = 0$) and $\frac{1}{x}$ is odd.

39. **(a)** $\lim_{x\to 1^-} f(x) = f(1)$

$$\lim_{x\to 1^-} (3 - x) = a(1)^2 + b(1)$$
$$2 = a + b$$

The relationship is $a + b = 2$.

39. Continued

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(3-(1+h))-2}{h} = \lim_{h\to 0^-}(-1) = -1$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{a(1+h)^2+b(1+h)-2}{h}$$
$$= \lim_{h\to 0^+} \frac{a+2ah+ah^2+b+bh-2}{h}$$
$$= \lim_{h\to 0^+} \frac{2ah+ah^2+bh+(a+b-2)}{h}$$
$$= \lim_{h\to 0^+}(2a+ah+b)$$
$$= 2a+b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$.
Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

40. True. See Theorem 1.

41. False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

42. B.

43. A. $\text{NDER}(f, x, a) = \dfrac{f(a+h)-f(a-h)}{2h}$

$$= \frac{\sqrt[3]{1.001-1}-\sqrt[3]{0.999-1}}{0.002} = 100$$

The symmetric difference quotient gets larger as h gets smaller, so $f'(1)$ is undefined.

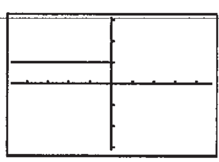
44. B. $\displaystyle\lim_{h\to 0^-} \frac{2(0+h)+1-(2(0)+1)}{h}$

$$= \lim_{h\to 0^-} \frac{2h}{h} = 2$$

45. C. $\displaystyle\lim_{h\to 0^+} \frac{(0+h)^2+1-(0^2+1)}{h}$

$$= \lim_{h\to 0^+} \frac{h^2}{h} = 0$$

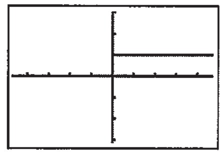
46. (a)



[−4.7, 4.7] by [−3.1, 3.1]

(b) You can use Trace to help see that the value of Y1 is 1 for every $x < 0$ and is 0 for every $x \ge 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0. \end{cases}$

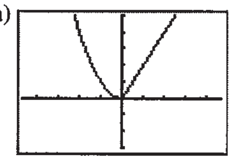
(c)



[−4.7, 4.7] by [−3.1, 3.1]

(d) You can use Trace to help see that the value of Y1 is 0 for every $x < 0$ and is 1 for every $x \ge 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0. \end{cases}$

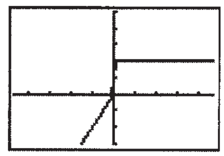
47. (a)



[−4.7, 4.7] by [−3, 5]

(b) See exercise 46.

(c)



[−4.7, 4.7] by [−3, 5]

(d) NDER (Y1, x, −0.1) = −0.1, NDER(Y1, x, 0) = 0.9995, NDER (Y1, x, 0.1) = 2.

48. (a) Note that $-|x| \le x\sin(1/x) \le |x|$ for all x except 0,

so $\displaystyle\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$ by the Sandwich Theorem.

Therefore, f is continuous at $x = 0$.

(b) $\dfrac{f(0+h)-f(0)}{h} = \dfrac{h\sin\frac{1}{h}-0}{h} = \sin\dfrac{1}{h}$

(c) The limit does not exist because $\sin\dfrac{1}{h}$ oscillates between −1 and 1 an infinite number of times arbitrarily close to $h = 0$ (that is, for h in any open interval containing 0).

(d) No, because the one-sided limits (as in part (c)) do not exist.

(e) $\dfrac{g(0+h)-g(0)}{h} = \dfrac{h^2\sin\left(\frac{1}{h}\right)-0}{h} = h\sin\dfrac{1}{h}$

As noted in part (a), the limit of this as h approaches zero is 0, so $g'(0) = 0$.

Section 3.3 Rules for Differentiation (pp. 116–126)

Quick Review 3.3

1. $(x^2-2)(x^{-1}+1) = x^2x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$

$= x + x^2 - 2x^{-1} - 2$