54. (a) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2 - a^2x)$
$= (2)^2 - a^2(2)$
$= 4 - 2a^2$

(b) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4 - 2x^2) = 4 - 2(2)^2$
$= 4 - 8 = -4$

(c) For $x \neq 2$, f is continuous. For $x = 2$, we have
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = -4$ as long as $a = \pm 2$.

55. (a) By inspection, $x^3 - 2x^2 + 1 = 0$ when $x = 1$. Use synthetic division to write $x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$; then use the quadratic formula to find the zeros of $x^2 - x - 1$ to be $\frac{1\pm\sqrt{5}}{2}$. The zeros of f are 1, $\frac{1+\sqrt{5}}{2}$, and $\frac{1-\sqrt{5}}{2}$.

(b) A right end-behavior model for f is $y = \frac{x^3}{x^2} = x$.

(c) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3 - 2x^2 + 1}{x^2 + 3} = +\infty$ and

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3 - 2x^2 + 1}{x^3 + 3x} = 1.$

Chapter 3
Derivatives

Section 3.1 Derivative of a Function (pp. 99–108)

Exploration 1 Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since y is measured in inches and x is measured in days, the derivative $\frac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the y-axis in Figure 3.3b.

2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of about 4 ³/₄ inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

5. The y-coordinate of point C gives the maximum depth of the water level in the ditch over the 7-day period, while the x-coordinate of C gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at C', indicating that this is when the water level stops rising and begins falling.

6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

Quick Review 3.1

1. $\lim_{h\to 0}\frac{(2+h)^2 - 4}{4} = \lim_{h\to 0}\frac{(4+4h+h^2)-4}{h}$
$= \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0}\frac{h(4+h)}{h}$
$= \lim_{h\to 0}(4+h)$
$= 4 + 0 = 4$

2. $\lim_{x\to 2^+}\frac{x+3}{2} = \frac{2+3}{2} = \frac{5}{2}$

3. Since $\frac{|y|}{y} = -1$ for $y < 0$, $\lim_{y\to 0^-}\frac{|y|}{y} = -1$.

4. $\lim_{x\to 4}\frac{2x-8}{\sqrt{x}-2} = \lim_{x\to 4}\frac{2(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$
$= \lim_{x\to 4} 2(\sqrt{x}+2) = 2(\sqrt{4}+2) = 8$

5. The vertex of the parabola is at (0, 1). The slope of the line through (0, 1) and another point $(h, h^2 + 1)$ on the parabola is $\frac{(h^2+1)-1}{h-0} = h$. Since $\lim_{h\to 0} h = 0$, the slope of the line tangent to the parabola at its vertex is 0.

6. Use the graph of f in the window [−6, 6] by [−4, 4] to find that (0, 2) is the coordinate of the high point and (2, −2) is the coordinate of the low point. Therefore, f is increasing on $(-\infty, 0]$ and $[2, \infty)$.

7. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-1)^2 = (1-1)^2 = 0$
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+2) = 1 + 2 = 3$

8. $\lim_{h\to 0^+} f(1+h) = \lim_{x\to 1^+} f(x) = 0$ (see Exercise 7).

9. No, the two one-sided limits are different (see Exercise 7).

10. No, f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist.

Section 3.1 Exercises

1. $f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$

$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2}{2(2+h)}-\frac{2+h}{2(2+h)}\right]=\lim_{h\to 0}\frac{1}{h}\left(\frac{h}{2(2+h)}\right)$

$=\lim_{h\to 0}\frac{1}{2(2+h)}=\frac{1}{4}$

2. $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{[(1+h)^2+4]-[1^2+4]}{h}$

$=\lim_{h\to 0}\frac{1+2h+h^2+4-5}{h}=\lim_{h\to 0}\frac{2h+h^2}{h}$

$=\lim_{h\to 0}\frac{h(2+h)}{h}$

$=\lim_{h\to 0}(2+h)$

$=2$

3. $f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$

$=\lim_{h\to 0}\frac{[3-(-1+h)^2]-[3-(-1)^2]}{h}$

$=\lim_{h\to 0}\frac{[3-(1-2h+h^2)]-(3-1)}{h}$

$=\lim_{h\to 0}\frac{2h-h^2}{h}$

$=\lim_{h\to 0}\frac{h(2-h)}{h}=\lim_{h\to 0}(2-h)=2$

4. $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{(h^3+h)-(0^3+0)}{h}$

$=\lim_{h\to 0}\frac{h(h^2+1)}{h}$

$=\lim_{h\to 0}(h^2+1)=1$

5. $f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

$=\lim_{x\to 2}\frac{1}{x-2}\left[\frac{2}{2x}-\frac{x}{2x}\right]$

$=\lim_{x\to 2}\frac{1}{x-2}\cdot\frac{2-x}{2x}$

$=\lim_{x\to 2}\frac{1}{x-2}\cdot\frac{-(x-2)}{2x}$

$=\lim_{x\to 2}\frac{-1}{2x}=-\frac{1}{4}$

6. $f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$

$=\lim_{x\to 1}\frac{(x^2+4)-(1^2+4)}{x-1}$

$=\lim_{x\to 1}\frac{x^2-1}{x-1}$

$=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$

$=\lim_{x\to 1}(x+1)=2$

7. $f'(3)=\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$

$=\lim_{x\to 3}\frac{\sqrt{x+1}-\sqrt{3+1}}{x-3}$

$=\lim_{x\to 3}\frac{(\sqrt{x+1}-2)}{(x-3)}\cdot\frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}$

$=\lim_{x\to 3}\frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$

$=\lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x+1}+2)}$

$=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}=\frac{1}{4}$

8. $f'(-1)=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$

$=\lim_{x\to -1}\frac{(2x+3)-(2(-1)+3)}{x+1}$

$=\lim_{x\to -1}\frac{2x+3-1}{x+1}=\lim_{x\to -1}\frac{2x+2}{x+1}$

$=\lim_{x\to -1}\frac{2(x+1)}{x+1}=\lim_{x\to -1}2=2$

9. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{[3(x+h)-12]-(3x-12)}{h}$

$=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3$

10. $\frac{dy}{dx}=\lim_{h\to 0}\frac{y(x+h)-y(x)}{h}$

$=\lim_{h\to 0}\frac{7(x+h)-7x}{h}$

$=\lim_{h\to 0}\frac{7h}{h}=\lim_{h\to 0}7=7$

11. Let $f(x)=x^2$

$\frac{d}{dx}(x^2)=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$

$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}(2x+h)=2x$

12. $\frac{d}{dx}f(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0}\frac{3(x+h)^2-3x^2}{h}$

$= \lim_{h\to0}\frac{3x^2+6xh+h^2-3x^2}{h}$

$= \lim_{h\to0}\frac{6xh+h^2}{h}$

$= \lim_{h\to0}(6x+h) = 6x$

13. The graph of $y = f_1(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so its derivative is negative for $x < 0$ and positive for $x > 0$. (b)

14. The graph of $y = f_2(x)$ is always increasing, so its derivative is always ≥ 0. (a)

15. The graph of $y = f_3(x)$ oscillates between increasing and decreasing, so its derivative oscillates between positive and negative. (d)

16. The graph of $y = f_4(x)$ is decreasing, then increasing, then decreasing, and then increasing, so its derivative is negative, then positive, then negative, and then positive. (c)

17. **(a)** The tangent line has slope 5 and passes through $(2, 3)$.

$y = 5(x-2)+3$

$y = 5x-7$

(b) The normal line has slope $-\frac{1}{5}$ and passes through $(2, 3)$.

$y = -\frac{1}{5}(x-2)+3$

$y = -\frac{1}{5}x+\frac{17}{5}$

18. $\frac{dy}{dx} = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0}\frac{[2(x+h)^2-13(x+h)+5]-(2x^2-13x+5)}{h}$

$= \lim_{h\to0}\frac{2x^2+4xh+2h^2-13x-13h+5-2x^2+13x-5}{h}$

$= \lim_{h\to0}\frac{4xh+2h^2-13h}{h}$

$= \lim_{h\to0}(4x+2h-13) = 4x-13$

At $x = 3$, $\frac{dy}{dx} = 4(3)-13 = -1$, so the tangent line has slope -1 and passes through $(3, y(3)) = (3, -16)$.

$y = -1(x-3)-16$

$y = -x-13$

19. Let $f(x) = x^3$.

$f'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to0}\frac{(1+h)^3-1^3}{h}$

$= \lim_{h\to0}\frac{1+3h+3h^2+h^3-1}{h}$

$= \lim_{h\to0}(3+3h+h^2) = 3$

(a) The tangent line has slope 3 and passes through $(1, 1)$. Its equation is $y = 3(x-1)+1$, or $y = 3x-2$.

(b) The normal line has slope $-\frac{1}{3}$ and passes through $(1, 1)$.

Its equation is $y = -\frac{1}{3}(x-1)+1$, or $y = -\frac{1}{3}x+\frac{4}{3}$.

20. Let $f(x) = \sqrt{x}$.

$f'(4) = \lim_{h\to0}\frac{f(4+h)-f(4)}{h}$

$= \lim_{h\to0}\frac{\sqrt{4+h}-\sqrt{4}}{h}$

$= \lim_{h\to0}\frac{(\sqrt{4+h}-2)}{h}\bullet\frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)}$

$= \lim_{h\to0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)}$

$= \lim_{h\to0}\frac{h}{h(\sqrt{4+h}+2)}$

$= \lim_{h\to0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$

(a) The tangent line has slope $\frac{1}{4}$ and passes through $(4, 2)$.

Its equation is $y = \frac{1}{4}x+1$.

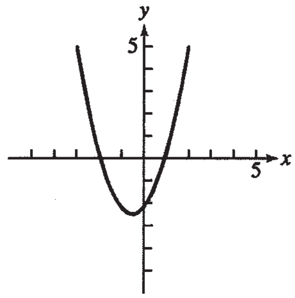
(b) The normal line has slope -4 and passes through $(4, 2)$. Its equation is $y = -4x+18$.

21. **(a)** The amount of daylight is increasing at the fastest rate when the slope of the graph is largest. This occurs about one-fourth of the way through the year, sometime around April 1. The rate at this time is approximately $\frac{4 \text{ hours}}{24 \text{ days}}$ or $\frac{1}{6}$ hour per day.

(b) Yes, the rate of change is zero when the tangent to the graph is horizontal. This occurs near the beginning of the year and halfway through the year, around January 1 and July 1.

(c) Positive: January 1 through July 1
Negative: July 1 through December 31

22. The slope of the given graph is zero at $x \approx -2$ and at $x \approx 1$, so the derivative graph includes $(-2, 0)$ and $(1, 0)$. The slopes at $x = -3$ and at $x = 2$ are about 5 and the slope at $x = -0.5$ is about -2.5, so the derivative graph includes $(-3, 5)$, $(2, 5)$, and $(-0.5, -2.5)$. Connecting the points smoothly, we obtain the graph shown.



23. (a) Using Figure 3.10a, the number of rabbits is largest after 40 days and smallest from about 130 to 200 days. Using Figure 3.10b, the derivative is 0 at these times.

(b) Using Figure 3.10b, the derivative is largest after 20 days and smallest after about 63 days. Using Figure 3.10a, there were 1700 and about 1300 rabbits, respectively, at these times.

24. Since the graph of $y = x \ln x - x$ is decreasing for $0 < x < 1$ and increasing for $x > 1$, its derivative is negative for $0 < x < 1$ and positive for $x > 1$. The only one of the given functions with this property is $y = \ln x$. Note also that $y = \ln x$ is undefined for $x < 0$, which further agrees with the given graph. (ii)

25. Each of the functions $y = \sin x$, $y = x$, $y = \sqrt{x}$ has the property that $y(0) = 0$ but the graph has nonzero slope (or undefined slope) at $x = 0$, so none of these functions can be its own derivative. The function $y = x^2$ is not its own derivative because $y(1) = 1$ but

$$y'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0}(2 + h) = 2.$$

This leaves only e^x, which can plausibly be its own derivative because both the function value and the slope increase from very small positive values to very large values as we move from left to right along the graph. (iv)

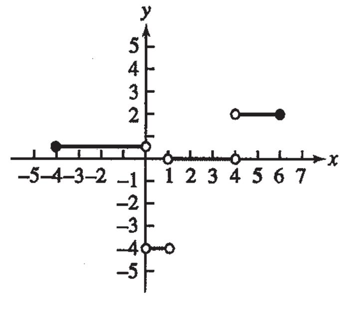
26. (a) The slope from $x = -4$ to $x = 0$ is $\frac{2-0}{0-(-4)} = \frac{1}{2}$.

The slope from $x = 0$ to $x = 1$ is $\frac{-2-2}{1-0} = -4$.

The slope from $x = 1$ to $x = 4$ is $\frac{-2-(-2)}{4-1} = 0$.

The slope from $x = 4$ to $x = 6$ is $\frac{2-(-2)}{6-4} = 2$.

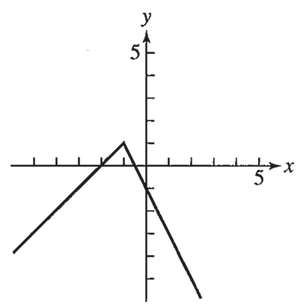
Note that the derivative is undefined at $x = 0$, $x = 1$, and $x = 4$. (The function is differentiable at $x = -4$ and at $x = 6$ because these are endpoints of the domain and the one-sided derivatives exist.) The graph of the derivative is shown.



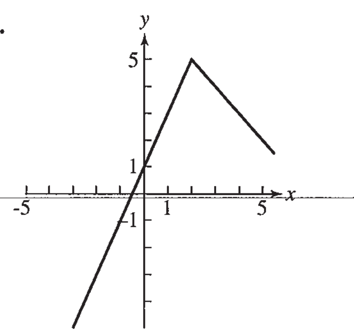
(b) $x = 0, 1, 4$

27. For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope -2 that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

Thus $f(x) = \begin{cases} x + 2, & x < -1 \\ -2x - 1, & x \geq -1 \end{cases}$



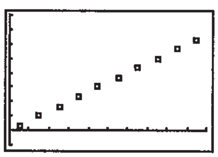
28.



29.

Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
0.5	$\frac{3.3-0}{1-0}=3.3$
1.5	$\frac{13.3-3.3}{2-1}=10.0$
2.5	$\frac{29.9-13.3}{3-2}=16.6$
3.5	$\frac{53.2-29.9}{4-3}=23.3$
4.5	$\frac{83.2-53.2}{5-4}=30.0$
5.5	$\frac{119.8-83.2}{6-5}=36.6$
6.5	$\frac{163.0-119.8}{7-6}=43.2$
7.5	$\frac{212.9-163.0}{8-7}=49.9$
8.5	$\frac{269.5-212.9}{9-8}=56.6$
9.5	$\frac{332.7-269.5}{10-9}=63.2$

A graph of the derivative data is shown.



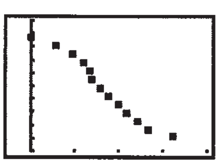
[0,10] by [−10,80]

(a) The derivative represents the speed of the skier.

(b) Since the distances are given in feet and the times are given in seconds, the units are feet per second.

(c) The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is $\frac{63.2-0}{9.5-0}\approx 6.65$. The equation of the derivative is approximately $D = 6.65t$.

30. (a)

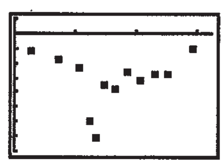


[−0.5, 4] by [700, 1700]

(b)

Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
$\frac{0.00+0.56}{2}=0.28$	$\frac{1512-1577}{0.56-0.00}\approx -116.07$
$\frac{0.56+0.92}{2}=0.74$	$\frac{1448-1512}{0.92-0.56}\approx -177.78$
$\frac{0.92+1.19}{2}=1.055$	$\frac{1384-1448}{1.19-0.92}\approx -237.04$
$\frac{1.19+1.30}{2}=1.245$	$\frac{1319-1384}{1.30-1.19}\approx -590.91$
$\frac{1.30+1.39}{2}=1.345$	$\frac{1255-1319}{1.39-1.30}\approx -711.11$
$\frac{1.39+1.57}{2}=1.48$	$\frac{1191-1255}{1.57-1.39}\approx -355.56$
$\frac{1.57+1.74}{2}=1.655$	$\frac{1126-1191}{1.74-1.57}\approx -382.35$
$\frac{1.74+1.98}{2}=1.86$	$\frac{1062-1126}{1.98-1.74}\approx -266.67$
$\frac{1.98+2.18}{2}=2.08$	$\frac{998-1062}{2.18-1.98}=-320.00$
$\frac{2.18+2.41}{2}=2.295$	$\frac{933-998}{2.41-2.18}\approx -282.61$
$\frac{2.41+2.64}{2}=2.525$	$\frac{869-933}{2.64-2.41}\approx -278.26$
$\frac{2.64+3.24}{2}=2.94$	$\frac{805-869}{3.24-2.64}\approx -106.67$

A graph of the derivative data is shown.



[0, 3.24] by [−800, 100]

(c) Since the elevation y is given in feet and the distance x down river is given in miles, the units of the gradiant are feet per mile.

(d) Since the elevation y is given in feet and the distance x downriver is given in miles, the units of the derivative $\frac{dy}{dx}$ are feet per mile.

(e) Look for the steepest part of the curve. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

(f) Look for the lowest point on the graph. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

31. We show that the right-hand derivative at 1 does not exist.

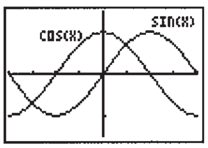
$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{3(1+h)-2-2}{h}$$
$$=\lim_{h\to 0^+}\frac{3h-1}{h}=-\infty$$

Does not exist.

32. We show that the right-hand derivative at 1 does not exist.

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{3(1+h)-(1)^3}{h}$$
$$=\lim_{h\to 0^+}\frac{2+3h}{h}=\lim_{h\to 0^+}\left(\frac{2}{h}+3\right)=\infty$$

33.



$[-\pi, \pi]$ by $[-1.5, 1.5]$

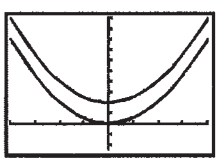
The cosine function could be the derivative of the sine function. The values of cosine are positive where sine is increasing, zero where sine has horizontal tangents, and negative where sine is decreasing.

34. $$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt{h}-\sqrt{0}}{h}=\lim_{h\to 0^+}\frac{\sqrt{h}}{h}$$
$$=\lim_{h\to 0^+}\frac{1}{\sqrt{h}}=\infty$$

Thus, the right-hand derivative at 0 does not exist.

35. Two parabolas are parallel if they have the same derivative at every value of x. This means that their tangent lines are parallel at each value of x.

Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.



[−4, 4] by [−5, 20]

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

36. True. $f'(x) = 2x + 1$

37. False. Let $f(x) = \frac{|x|}{x}$. The left hand derivative at $x = 0$ is -1 and the right hand derivative at $x = 0$ is 1. $f'(0)$ does not exist.

38. C. $$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$
$$=\lim_{h\to 0}\frac{[4-3(-1+h)]-[4-3(-1)]}{h}$$
$$=\lim_{h\to 0}\frac{4+3-3h-7}{h}=\lim_{h\to 0}\frac{-3h}{h}=\lim_{h\to 0}-3=-3$$

39. A. $$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{[1-3(1+h)^2]-[1-3(1)^2]}{h}$$
$$=\lim_{h\to 0}\frac{-6h-3h^2}{h}=\lim_{h\to 0}\frac{h(-6-3h)}{h}$$
$$=\lim_{h\to 0}(-6-3h)=-6$$

40. B. $$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}$$
$$=\lim_{h\to 0^-}\frac{(h^2-1)-(-1)}{h}=\lim_{h\to 0^-}\frac{h^2}{h}$$
$$=\lim_{h\to 0^-}h=0$$

41. C. $$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$$
$$=\lim_{h\to 0^+}\frac{(2h-1)-(-1)}{h}=\lim_{h\to 0^+}\frac{2h}{h}$$
$$=\lim_{h\to 0^+}2=2$$

42. **(a)** $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$=\lim_{h\to 0}(2x+h)=2x$$

(b) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{2(x+h)-2x}{h}=\lim_{h\to 0}2=2$$

(c) $\lim_{x\to 1^-}f'(x)=\lim_{x\to 1^-}2x=2(1)=2$

(d) $\lim_{x\to 1^+}f'(x)=\lim_{x\to 1^+}2=2$

(e) Yes, the one-sided limits exist and are the same, so $\lim_{x\to 1}f'(x)=2$.

(f) $$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{(1+h)^2-1^2}{h}$$
$$=\lim_{h\to 0^-}\frac{1+2h+h^2-1}{h}$$
$$=\lim_{h\to 0^-}\frac{2h+h^2}{h}=\lim_{h\to 0^-}\frac{h(2+h)}{h}$$
$$=\lim_{h\to 0^-}(2+h)=2$$

(g) $$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{2(1+h)-1^2}{h}$$
$$=\lim_{h\to 0^+}\frac{1+2h}{h}=\lim_{h\to 0^+}\left(\frac{1}{h}+2\right)=\infty$$

The right-hand derivative does not exist.

(h) It does not exist because the right-hand derivative does not exist.

43. **(e)** The y-intercept of the derivative is $b - a$.

44. Since the function must be continuous at $x = 1$, we have $\lim_{x\to 1^+}(3x+k) = f(1) = 1$, so $3+k=1$, or $k=-2$.

This gives $f(x) = \begin{cases} x^3, & x \le 1 \\ 3x-2, & x > 1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^3-(1)^3}{h}$$
$$= \lim_{h\to 0^-} \frac{3h+3h^2+h^3}{h}$$
$$= \lim_{h\to 0^-} (3+3h+h^2) = 3$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{[3(1+h)-2]-(1)^3]}{h}$$
$$= \lim_{h\to 0^+} \frac{(1+3h)-1}{h} = \lim_{h\to 0^+} 3 = 3$$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

45. **(a)** $1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.992$

Alternate method: $\frac{{}_{365}P_3}{365^3} \approx 0.992$

(b) Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

(c) Let P represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is

$(1-P)\left(\frac{362}{365}\right)$, so the probability of a shared birthday is

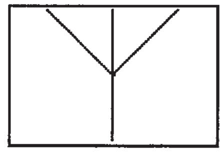
$1-(1-P)\left(\frac{362}{365}\right) \approx 0.016.$

(d) No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

Section 3.2 Differentiability (pp. 109–115)

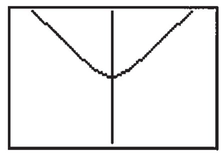
Exploration 1 Zooming in to "See" Differentiability

1. Zooming in on the graph of f at the point (0, 1) always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.



[−0.25, 0.25] by [0.836, 1.164]

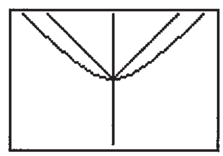
2. Zooming in on the graph of g at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window.
[−4, 4] by [−1.624, 3.624].



[−0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.



[−0.03125, 0.03125] by [0.9795, 1.0205]

Exploration 2 Looking at the Symmetric Difference Quotient Analytically

1. $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^2-10^2}{0.01} = 20.01$

$f'(10) = 2 \cdot 10 = 20$

The difference quotient is 0.01 away from $f'(10)$.

2. $\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^2-(9.99)^2}{0.02} = 20$

The symmetric difference quotient exactly equals $f'(10)$.

3. $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^3-10^3}{0.01} = 300.3001.$

$f'(10) = 3 \cdot 10^2 = 300$

The difference quotient is 0.3001 away from $f'(10)$.

$\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^3-(9.99)^3}{0.02} = 300.0001.$ The symmetric difference quotient is 0.0001 away from $f'(10)$.